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Unipotent group actions on affine varieties

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ABSTRACT

Algebraic actions of unipotent groups U on affine k-varieties X (k is an algebraically closed field of characteristic 0) for which the algebraic quotient $X//U$ has small dimension are considered. In case X is factorial, $O(X)^* = k^*$, and $X//U$ is one-dimensional, it is shown that $O(X)^U = k[f]$, and if some point in X has trivial isotropy, then X is U equivariantly isomorphic to $U \times A^1(k)$. The main results are given distinct geometric and algebraic proofs. Links to the Abhyankar–Sathaye conjecture and a new equivalent formulation of the Sathaye conjecture are made.

1. Preliminaries and introduction

Throughout, k will denote a field of characteristic zero, $k^{[n]}$ the polynomial ring in n variables over k, and U a unipotent algebraic group over k. Our primary interest is in algebraic actions of such U on quasiaffine k-varieties X (equivalently on their rings $\mathcal{O}(X)$ of globally defined regular functions). An algebraic action of the one-dimensional unipotent group $\mathbb{G}_a(k) = (k, +)$ (which will be denoted by $\mathbb{G}_a$ when the base field is clear from the context) is conveniently described through the action of a locally nilpotent derivation D of $\mathcal{O}(X)$. Specifically, for $u \in \mathbb{G}_a$, we have the automorphism u^* acting on $\mathcal{O}(X)$ and it is well known (see for example [1, pp. 16–17]) that there exists a unique locally nilpotent derivation $D : \mathcal{O}(X) \to \mathcal{O}(X)$ such that $u^* = \exp(uD)$. (One can obtain D by taking $D(f) = \frac{u^* f - f}{u}|_{u=0}$.) Similarly, if $\mathbb{G}_a^n$ acts on X, then we have for each component $\mathbb{G}_a$-action

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a locally nilpotent derivation D_i, and for each element $u=(u_1,\dots,u_n)\in\mathbb{G}_a^n$ we have the derivation $D:=u_1D_1+\cdots+u_nD_n$. If the action is faithful, there is a canonical isomorphism of $\mathrm{Lie}(\mathbb{G}_a^n)$ with $kD_1+\cdots+kD_n$. In this case, the D_i commute.

The situation is similar for a general unipotent group action $U\times X\to X$. Because the action is algebraic, each $f\in\mathcal{O}(X)$ is contained in a finite-dimensional U stable subspace V_f on which U acts by linear transformations. Since U is unipotent, for each $u\in U$, u^*-id is nilpotent on V_f, so that $\ln(u)(g)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j}(u^*-id)^j(g)$ is a finite sum for all $g\in V_f$. By expanding the (finite) sum $\sigma_u(t):=\exp(t\ln u)(f)=\sum\frac{t^n}{n!}(\ln u)^n(f)$ one checks that a $\mathbb{G}_a$ action σ_u on $k^{[n]}$ is obtained and that $\ln(u)(f)=\frac{\sigma_u(t)f-f}{t}|_{t=0}$. Thus $D_u:=\ln(u)$ defines a (locally nilpotent) derivation of $\mathcal{O}(X)$ and $u^*=\exp(D_u)$. If the action is faithful, i.e. $U\to Aut(X)$ is injective, there is a canonical isomorphism of $\mathrm{Lie}(U)$ with $\{D_u\mid u\in U\}$. In fact, $\mathrm{Lie}(U)=kD_1+\cdots+kD_m$ ($m=\dim(U)$) for some locally nilpotent derivations D_i. In general the D_i do not commute. In fact, all of them commute if and only if $U=\mathbb{G}_a^m$.

Two useful facts about unipotent group actions on quasiaffine varieties X can be immediately deduced from these observations:

(1) Because each $u\in U$ acts via a locally nilpotent derivation of $O(X)$, the ring of invariants $O(X)^U$ is the intersection of the kernels of locally nilpotent derivations.
(2) Since kernels of locally nilpotent derivations D are factorially closed, meaning that $ab\in\ker D$ implies both a and b lie in $\ker D$, their intersection is too, i.e. $O(X)^U$ is factorially closed. In particular if $O(X)$ is a UFD then so is $O(X)^U$.

The term factorial for a quasiaffine variety X is used here to mean a quasiaffine variety X for which $O(X)$ is a UFD. This is a more restrictive meaning than having all local rings be UFDs. Given a locally nilpotent derivation D on the k-algebra A, an element $a\in A$ is called a preslice for D if $D(a)\neq 0=D^2(a)$. An element $s\in A$ is called a slice if $D(s)=1$. If a slice exists, $A=(\ker D)[s]$ [12].

We will use the fact that U is a special group in the sense of Serre. This means that a U action which is locally trivial for the étale topology is locally trivial for the Zariski topology. If G is a group acting on a variety X, we denote by $X//G$ the algebraic quotient $X//G:=\mathbf{Spec}\,\mathcal{O}(X)^G$ and by X/G the geometric quotient (when it exists). By a free action we mean an action for which the isotropy subgroup of each element consists only of the identity. (A free action is faithful.) A useful classical reference for results on algebraic actions of unipotent groups (e.g. that all orbits are closed) is [13]. Throughout $\mathbb{A}^n(k)$ (resp. $\mathbb{P}^n(k)$) denote n-dimensional affine (resp. projective) space over the field k, and the (k) will be omitted when the base is clear from the context.

The paper is organized as follows: Section 2 contains some examples which illustrate the main results and clarify their hypotheses. The main results are proved in Section 3 from a geometric perspective, and Section 4 gives them an algebraic interpretation. (The algebraic and geometric viewpoint both have their merits: the geometric viewpoint lends itself to possible generalizations, while the algebraic proofs are constructive and can be more easily used in algorithms.) In Section 5 we elaborate on some implications of the main results for the Sathaye conjecture, and on the motivation for studying this problem.

2. Examples

The following examples illustrate some of the observations made in the introduction and are valuable in various parts of the subsequent development. For a k algebra A, the notation $DER(A)$ refers to the A module of k derivations of A.

Example 1. Let $X=k^3$, and $U:=\{u_{a,b,c}\mid a,b,c\in k\}$ where

$$u_{a,b,c}:=\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}$$

a unipotent group acting by $u_{a,b,c}(x,y,z) = (x+a, y+az+b, z+c)$ (which indeed is an algebraic action). For each $(a,b,c) \in k^3$ we thus have an automorphism, and its associated derivation on $k[X,Y,Z]$ is $D_{a,b,c} = a\partial_X + (aZ + b - \frac{ac}{2})\partial_Y + c\partial_Z$. Set $D_1 = \partial_Y$, $D_2 := \partial_X + Z\partial_Y$, $D_3 = \partial_Z$. As a Lie algebra $\mathrm{Lie}(U)$ is generated by D_1, D_2, D_3. One checks that D_1 commutes with D_2, D_3, but $[D_2, D_3] = D_1$. However, restricted to $k[X,Y,Z]^{D_1} = k[X,Z]$, D_2 and D_3 do commute, as they coincide with the derivations ∂_X and ∂_Z. Furthermore, as a k vector space $\mathrm{Lie}(U)$ has basis $\partial_X, \partial_Y, \partial_Z$.

Example 2. Let $\mathcal{O}(X) = A = k[X,Y,Z]$, and $D_1 = Z\partial_X$, $D_2 = \partial_Y$. These locally nilpotent derivations generate a $U = (\mathbb{G}_a)^2$-action on k^3 given by $(a,b)\cdot(x,y,z) \to (x+az, y+b, z)$. Now $k[Z] = A^{D_1,D_2} = \mathcal{O}(X/U)$. D_1, D_2 are linearly independent over $k[Z]$. When calculating modulo $Z-\alpha$ where $\alpha \in k$, we notice that $D_1 \bmod (Z-\alpha)$, $D_2 \bmod (Z-\alpha)$ are linearly independent over $A/(Z-\alpha)$ except when $\alpha = 0$. However, defining $\mathcal{M} := (\mathrm{Lie}(U) \otimes k(Z)) \cap DER(A) = (k(Z)D_1 + k(Z)D_2) \cap DER(A)$ we see that $\mathcal{M} = k[Z]\partial_X + k[Z]\partial_Y$. The derivations ∂_X, ∂_Y are linearly independent modulo each $Z-\alpha$. And for each $\alpha \in k$, we have $A/(Z-\alpha) \cong k^{[2]}$.

Example 3. Let $P := X^2Y + X + Z^2 + T^3$, $\mathcal{X} := \{(x,y,z,t) \mid P(x,y,z,t) = 0\}$. Let $A := k[x,y,z,t] := k[X,Y,Z,T]/(P) = \mathcal{O}(\mathcal{X})$. The commuting locally nilpotent derivations $2Z\partial_Y - X^2\partial_Z$, $3T^2\partial_Y - X^2\partial_T$ on $k[X,Y,Z,T]$ map P to zero, and hence induce derivations D_1, D_2 on A. They are linearly independent over $A^{D_1,D_2} = k[X]$ and since they commute, induce a $(\mathcal{G}_a)^2$-action on $\mathcal{X}$. Modulo $X-\alpha$, D_1, D_2 are linearly independent, except when $\alpha = 0$. Now defining $\mathcal{M} := (\mathrm{Lie}(U) \otimes k(X)) \cap DER(A) = k[X]D_1 + k[X]D_2 = \mathrm{Lie}(U) \otimes k[X]$, we see that $\mathcal{M}$ modulo $X-\alpha$ is a k-module of dimension 2 except when $\alpha = 0$, when it is of dimension 1. Also, $A/(X-\alpha) \cong k^{[2]}$ except when $\alpha = 0$, when it is isomorphic to $R[X]$ where $R = k[Z,T]/(Z^2+T^3)$.

Example 4. The $U = \mathbb{G}_a \times \mathbb{G}_a$ action on $\mathbb{A}^2(k)$ given by

$$U \times \mathbb{A}^2 \ni \big((s,t),(x,y)\big) \mapsto (x, y+t+sx) \in \mathbb{A}^2$$

is faithful and fixed point free. However every point in $\mathbb{A}^2$ has a nontrivial isotropy subgroup. If $x \neq 0$, then $((s,-sx),(x,y)) \mapsto (x,y)$ and $((s,0),(0,y)) \mapsto (0,y)$.

3. Main results

The following lemma is useful in a number of places. In this section we take k to be algebraically closed (and of characteristic 0).

Lemma 1. *Let U be a unipotent algebraic group acting algebraically on a factorial quasiaffine variety X of dimension n satisfying $\mathcal{O}(X)$ finitely generated as a k-algebra and $\mathcal{O}(X)^* = k^*$. If the action is not transitive and some point $x \in X$ has orbit of dimension $n-1$, then $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$.*

Proof. There is a Zariski open subset V of X for which the geometric quotient V/U exists as a variety. Since $n-1$ is the maximum orbit dimension on X, this is the dimension of all U orbits on V, and the transcendence degree of the quotient field K of $\mathcal{O}(V/U)$ is equal to 1 [13,14]. Since $K = qf(\mathcal{O}(X)^U)$ and

$$\mathcal{O}(X)^U = \mathcal{O}(X) \cap K,$$

a theorem of Zariski [10] yields that $\mathcal{O}(X)^U$ is finitely generated over k. Although it is well known (e.g. [9]) that since $\mathcal{O}(X)^U$ is a UFD, it is in fact a polynomial ring in one variable over k, an argument is sketched for the convenience of the reader: Set $Y := \mathbf{Spec}\,\mathcal{O}(X)^U$ and view Y as an open subset of a desingularization of a projective closure $\tilde{Y}$. By factoriality, any pair P, Q of points in Y are linearly equivalent and therefore give rise to an embedding $\tilde{Y} \to \mathbb{P}^1(k)$ [5, Chapter II, Section 7]. Thus Y is isomorphic to an affine open subset of $\mathbb{P}^1(k)$, hence to the complement in $\mathbb{A}^1$ of a finite subset. But $\mathcal{O}(Y)^* \subset \mathcal{O}(X)^* = k^*$ implies that $Y \cong \mathbb{A}^1$, i.e. $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$. $\square$

Remark 1. The key issue in the argument above is that the genus of K is equal to 0. A purely algebraic proof of this fact can be found in [2].

3.1. Unipotent actions having zero-dimensional quotient

Theorem 1. *Let U be an n-dimensional unipotent group acting faithfully on an affine n-dimensional variety X satisfying $\mathcal{O}(X)^* = k^*$. Then $X \cong \mathbb{A}^n$ if one of the following two conditions holds*:

(a) *some $x \in X$ has trivial isotropy subgroup, or*
(b) *$n = 2$, X is factorial, and U acts without fixed points.*

In case (a) *the action is transitive.*

Proof. In case (a) there is an open affine subset V of X on which U acts without fixed points. Since U has the same dimension as V, $V//U$ is zero-dimensional, hence $\mathcal{O}(V//U)$ is a field. This field contains k, and its units are contained in $\mathcal{O}(X)^* = k^*$, hence $\mathcal{O}(V//U) = k$. It follows that there exists an open set V' of X for which $V'/U \cong \mathbf{Spec}\, k$. Thus $V' \cong U$ as a variety, and therefore $V' \cong \mathbb{A}^n$. If $v \in V'$, then $Uv = V'$. Since U is unipotent, all orbits are closed, hence V' is closed in X. Since it is of dimension n, and X is irreducible of dimension n, we have that $V' = X$.

In case (b) X is acted on nontrivially by $\mathbb{G}_a(k)$ via $\exp(D_u)$ for some $u \in U$. Because X is factorial, $D_u = g\delta$ with δ locally nilpotent, $g \in \ker D_u = \ker\delta$, and $\exp(\delta)$ acting freely on X [4]. As in the proof of Lemma 1, the ring of invariants $\mathcal{O}(X)^{\mathbb{G}_a}(= \ker(\delta))$ for this $\mathbb{G}_a$ action is equal to $k[f]$ for some $f \in \mathcal{O}(X)$. On the other hand, free $\mathbb{G}_a$ actions on factorial affine surfaces are known to be equivariantly trivial in the sense that $X \cong_{\mathbb{G}_a} X/\mathbb{G}_a \times \mathbb{G}_a \cong \mathbf{Spec}\, \mathcal{O}(X)^{\mathbb{G}_a} \times \mathbb{G}_a$ where $\mathbb{G}_a$ acts trivially on the first factor and by addition on the second [3]. Thus $X \cong \mathbb{A}^2$. $\square$

Example 4 of the previous section illustrates case (b).

3.2. Unipotent actions having one-dimensional quotient

The following theorem is the main result of this paper.

Theorem 2 *(Main theorem). Let U be a unipotent algebraic group of dimension n, acting on X, a factorial variety of dimension $n+1$ satisfying $\mathcal{O}(X)^* = k^*$.*

(1) *If at least one $x \in X$ has trivial stabilizer then $\mathcal{O}(X)^U = \mathcal{O}(X//U) = k[f]$. Furthermore, $f^{-1}(\lambda) \cong \mathbb{A}^n$ for all but finitely many $\lambda \in k$.*
(2) *If U acts freely, then X is U-isomorphic to $U \times k$. In particular, $X \simeq \mathbb{A}^n$ and f is a coordinate.*

An important example to keep in mind is Example 1, as this satisfies (1) but not (2). (There $U = \mathbb{G}_a^2$.)

Proof of Theorem 2.

Claim 1. $\mathcal{O}(X)^U = k[f]$.

Proof. This follows from Lemma 1 and proves the first assertion. For the remainder assume that U acts freely. $\square$

Claim 2. $f : X \to \mathbb{A}^1$ *is surjective and has fibers isomorphic to U. The fibers are the U-orbits.*

Proof. The fibers $f^{-1}(\lambda)$ are the zero loci of the irreducible $f-\lambda$, and are invariant under U. Since U acts freely on each fiber and orbits of unipotent group actions are closed, we see that the f fibers are exactly the U orbits in X. Thus f is a U-fibration (and, as the underlying variety of U is $\mathbb{A}^n$, an $\mathbb{A}^n$-fibration). □

Claim 3. *X is smooth.*

Proof. The set of singular points of X, denoted by X_{sing} is U-stable, hence it is a union of U-orbits. The U-orbits are the zero sets $f-\lambda$, hence of codimension 1. So X_{sing} is of codimension 1 or empty. But X is factorial, so in particular normal, which implies that X_{sing}, is of codimension at least 2. This means that X_{sing} can only be empty. □

Claim 4. *f is smooth.*

Proof. All fibers of f are isomorphic to U, hence to $\mathbb{A}^n$, by Claim 2. Thus the fibers of f are geometrically regular of dimension n. Since X is smooth, f is flat, and [5, Proposition 10.2] yields that f is smooth. □

Claim 5. *$X \times_f X$ is smooth.*

Proof. $X \times_f X$ is smooth since it is a base extension of the smooth X by the smooth morphism f. □

Claim 6. *$g : U \times X \to X \times_f X$ given by $(u, x) \mapsto (x, ux)$ is an isomorphism.*

Proof. The map g restricted to $U \times f^{-1}(\lambda)$ is a bijection onto $\{(x, y) \mid f(x) = f(y) = \lambda\}$. Taking the union over $\lambda \in \mathbb{A}^1$, we get that g is a bijection. Since both $U \times X$ and $X \times_f X$ are smooth, the characteristic of k is zero, and g is a bijection on geometric points g is also birational. Zariski's Main Theorem implies that g is an open immersion and therefore an isomorphism since it is bijective. □

Now we are ready to prove the theorem. Using Definition 0.10, p. 16 of [11], and the fact (4) that f is smooth, together with (6), yields that $f : X \to \mathbb{A}^1$ is an étale principal U-bundle and therefore a Zariski locally trivial principal U bundle as U is special. Such bundles are classified by the cohomology set $H^1_{et}(\mathbb{A}^1, U)$, which is trivial because U is unipotent and $\mathbb{A}^1$ affine. (For $U = \mathbb{G}_a$ and any affine Z this follows from $H^1_{et}(Z, \mathbb{G}_a) \cong H^1(Z, \mathcal{O}_Z) = 0$. For general U argue by induction on n as follows: take a decreasing chain of normal subgroups $U = U_0 \rhd U_1 \rhd \cdots \rhd U_r = \{1\}$ with $U_i/U_{i+1} \cong \mathbb{G}_a$. Then apply induction based on the exact sequence [8, Chapter III, Proposition 4.5]

$$H^1_{et}(Z, U_{i+1}) \to H^1_{et}(Z, U_i) \to H^1_{et}(Z, U_i/U_{i+1})$$

to obtain the triviality of $H^1_{et}(\mathbb{Z}, U)$.) Thus the bundle $f : X \to \mathbb{A}^1$ is trivial, which means that $X \cong U \times \mathbb{A}^1$. □

Remark 2. One can avoid the use of the étale topology by applying a "Seshadri cover" [15]. One constructs a variety Z finite over X, necessarily affine, to which the U action extends so that:

(1) $k(Z)/k(X)$ is Galois. Denote the Galois group by Γ.
(2) The Γ and U actions commute on Z.
(3) The U action on Z is Zariski locally trivial and, because the action on X is proper by Claim 6.
(4) $Y \equiv Z/U$ exists as a separated scheme of dimension 1, hence is a curve, and affine because of the existence of nonconstant globally defined regular functions, namely $\mathcal{O}(Z)^U$.
(5) $\mathcal{O}(X)^U \cong \mathcal{O}(Y)^\Gamma$, and $X//U \cong X/U \cong Y/\Gamma$ shows that $X \to X/U$ is Zariski locally trivial.

4. Algebraic version

4.1. Unipotent actions having zero-dimensional kernel

Let X be a quasiaffine variety, and U an algebraic group acting on X. We write $A := \mathcal{O}(X)$ and denote by $\mathfrak{u}$ the Lie algebra of U. In this section, we will make the following assumptions:

(P) (a) X and U are of dimension n.
(b) There is a point $x \in X$ such that $\mathrm{stab}(x) = \{e\}$.
(c) $\mathcal{O}(X)^* = k^*$.

Definition 1. Assume (P). We say that $D_1, \ldots, D_n$ is a triangular basis of $\mathfrak{u}$ (with respect to the action on X) if

(1) $\mathfrak{u} = kD_1 \oplus kD_2 \oplus \cdots \oplus kD_n$, and
(2) with subalgebras A_i of A given by $A_1 := A$, $A_i := A^{D_1} \cap \cdots \cap A^{D_{i-1}}$, the restriction of D_i to A_i commutes with the restrictions of $D_{i+1}, \ldots, D_n$.

For a triangular basis, it is clear that $D_j(A_j) \subseteq A_j$ for each j.

If U is unipotent then the existence of a triangular basis is a consequence of the Lie–Kolchin theorem. Indeed, the Lie algebra $\mathfrak{u}$ of U is isomorphic to a Lie subalgebra of the full Lie algebra of upper triangular matrices over k. In particular $\mathfrak{u}$ has a basis $D_1, \ldots, D_n$ satisfying $[D_i, D_j] \in \mathit{span}\{D_1, \ldots, D_{\min\{i,j\}-1}\}$. By definition of the A_i this basis is triangular with respect to the action and D_1 is in the center of $\mathfrak{u}$.

Proposition 1. *Assume* (P) *and U unipotent. Then $A \cong k[s_1, \ldots, s_n] = k^{[n]}$ where $D_i(s_i) = 1$, and $D_i(s_j) = 0$ if $j > i$.*

Proof. We proceed by induction $n = \dim \mathfrak{u}$. If $n = 1$, then we have one nonzero locally nilpotent derivation on a dimension one k-algebra domain A satisfying $A^* = k^*$. It is well known that this means that $A \cong k[x]$ and the derivation is simply ∂_x. Suppose the theorem is proved for $n-1$. Let $D_1, D_2, \ldots, D_n$ be a triangular basis for $\mathfrak{u}$. Restricting to A^{D_1} and noting that D_1 is in the center of $\mathfrak{u}$, we have an action of the Lie algebra $\mathfrak{u}/kD_1$ which has the triangular basis $k\overline{D_2} + \cdots + k\overline{D_n}$ ($\overline{D_i}$ denotes residue class modulo kD_1). By construction $\overline{D}_i(a) := D_i(a)$ is well defined, and by induction we find $s_2, \ldots, s_n \in A^{D_1}$ satisfying $D_i(s_i) = 1, D_i(s_j) = 0$ if $j > i \geqslant 2$.

Next we consider a preslice $p \in A$ such that $D_1(p) = q, D_1(q) = 0$, i.e. $q = q(s_2, \ldots, s_n)$. We pick p in such a way that q is of lowest possible lexicographic degree with respect to $s_2 \gg s_3 \gg \cdots \gg s_n$. Now $D_1(D_2(p)) = D_2D_1(p) = D_2(q)$. Restricted to $k[s_2, \ldots, s_n]$, $D_2 = \partial_{s_2}$, so $D_2(q)$ is of lower s_2-degree than q. Unless $D_2(q) = 0$, we get a contradiction with the degree requirements of q, as $D_2(p)$ would be a "better" preslice having a lower degree derivative. Thus, $q \in k[s_3, \ldots, s_n]$. Using the same argument for D_3, D_4 etc. we get that $q \in k^*$. Hence, p is in fact a slice. $\square$

4.2. Unipotent actions having one-dimensional quotient

With the same notations as in the previous section, we also denote the ring of U invariants in A by A^U and $\mathbf{Spec}\, A^U$ by $X//U$. Note that $A^U = \{a \in A \mid D(a) = 0 \text{ for all } D \in \mathfrak{u}\}$. If U is unipotent and $D_1, \ldots, D_n$ is a triangular basis of $\mathfrak{u}$, we again write $A_1 := A$, $A_{i+1} = A_i \cap A^{D_i}$, noting that $A^U = A_n$. In this section we consider the conditions:

(Q1) U is a unipotent algebraic group of dimension n acting on an affine variety X of dimension $n+1$ with $A^* = k^*$, and

(Q) $A^U = k[f]$ for some irreducible $f \in A \backslash k$.

Remark 3. According to Lemma 1, condition (Q1) along with the assumption that X is factorial and the existence of a point $x \in X$ with $\operatorname{stab}(x) = \{e\}$, implies that (Q) holds.

Notation 1. Assuming (Q), let $\alpha \in k$. Set $\overline{A} := A/(f-\alpha)$ and write $\bar{a}$ for the residue class of a in $\overline{A}$ and $\overline{D}$ for the derivation induced by $D \in \mathfrak{u}$ on $\overline{A}$.

Our goal is to prove the following constructively:

Theorem 3. *Assume* (Q1) *and* (Q)*. Let $D_1, \dots, D_n$ be a triangular basis of* $\mathfrak{u}$*.*

(1) *For $\alpha \in k$:*
   (a) *If $\overline{D_1}, \dots, \overline{D_n}$ are independent over $A/(f-\alpha)$, then*

$$A/(f-\alpha) \cong k^{[n]}.$$

   (b) *There are only finitely many α for which $\overline{D_1}, \dots, \overline{D_n}$ are dependent over $A/(f-\alpha)$.*

(2) *In the case that $\overline{D_1}, \dots, \overline{D_n}$ are independent over $A/(f-\alpha)$ for each $\alpha \in k$, then there are $s_1, \dots, s_n \in A$ with $A = k[s_1, \dots, s_n, f]$, hence A is isomorphic to a polynomial ring in $n+1$ variables* (*and f is a coordinate*)*.*

Definition 2. Assume (Q1) and (Q), and a triangular basis $D_1, \dots, D_n$ of $\mathfrak{u}$. Define

$$\mathcal{P}_i := \{p \in A \mid D_i(p) \in k[f],\ D_j(p) = 0 \text{ if } j < i\}$$

and

$$\mathcal{J}_i := D_i(\mathcal{P}_i) \subseteq k[f].$$

Thus $\mathcal{P}_i$ is the set of "preslices" of D_i that are compatible with the triangular basis $D_1, \dots, D_n$.

Lemma 2. *There exist $p_i \in \mathcal{P}_i \backslash \{0\}$, $p_i \in A_i$, and $q_i \in k^{[1]} \backslash \{0\}$ such that $\mathcal{J}_i = q_i(f)k[f]$ and $D_i(p_i) = q_i$.*

Proof. First note that $\mathcal{J}_i$ is not empty, as Theorem 1 applied to $A(f) := A \otimes k(f)$ gives an $s_i \in A(f)$ which satisfies $D_i(s_i) = 1$, $D_j(s_i) = 0$ if $j < i$. Multiplying s_i by a suitable element of $k[f]$ gives a nonzero element $r(f)s_i$ of $\mathcal{P}_i$, and $D_i(r(f)s_i) = r(f)$. Because $k[f] = \bigcap \ker(D_i)$, $\mathcal{P}_i$ is a $k[f]$-module, and therefore $\mathcal{J}_i$ is an ideal of $k[f]$. This means that $\mathcal{J}_i$ is a principal ideal, and we take for q_i a generator (and $p_i \in D_i^{-1}(q_i)$). Since $D_j(p_i) = 0$ if $j < i$, we have $p_i \in A_i$. □

Corollary 1. *The p_i, $1 \leqslant i \leqslant n$, are algebraically independent over k.*

Proof. The s_i are certainly algebraically independent, and $p_i \in k[f]s_i$. □

Lemma 3. *Assume* (Q)*, and take p_i, q_i as in Lemma* 2*. Then the D_i are linearly dependent modulo $f-\alpha$ if and only if $q_i(\alpha) = 0$ for some i.*

Proof. ($\Rightarrow$): Suppose that $0 \neq D := g_1 D_1 + \cdots + g_n D_n$ satisfies $\overline{D} = 0$ where $g_i \in A$, and not all $\bar{g}_i = \bar{0}$. Let i be the highest such that $\bar{g}_i \neq \bar{0}$. Then $0 = \overline{D(p_i)} = \bar{g}_i \bar{D} \bar{p}_i = \bar{g}_i \overline{q_i(f)}$. Since $\bar{A}$ is a domain, $q_i(\alpha) = \overline{q_i(f)} = 0$.

($\Leftarrow$): Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $\overline{D_i}$ are linearly dependent over $A/(f-\alpha)$. Consider $\bar{D}_i$ restricted to $\bar{A}_i$. If $j > i$ then $\bar{D}_i(p_j) = \overline{D_i(p_j)} = \bar{0}$. Furthermore $\bar{D}_i(\bar{p}_i) = \bar{q}_i(f) = q(\alpha) = 0$. Hence, $\bar{D}_i$ is zero if restricted to $k[\bar{p}_i, \ldots, \bar{p}_n]$. But since this is of transcendence degree n, it follows that $\bar{D}_i = 0$ on $\bar{A}_i$. Reversing the argument yields the linear dependence of the $\bar{D}_i$. $\square$

Proof of Theorem 3. *Part 1*: If $\bar{D}_1, \ldots, \bar{D}_n$ are independent, then Proposition 1 yields that $\bar{A} \cong k^{[n]}$. Lemma 3 states that for any point α outside the zero set of $q_1 q_2 \cdots q_n$ we have $A/(f-\alpha) \cong k^{[n]}$. This zero set is either all of k or finite, yielding part 1.

Part 2: Lemma 3 tells us directly that for each $1 \leqslant i \leqslant n$ and $\alpha \in k$, we have $q_i(\alpha) \neq 0$. But this means that the $q_i \in k^*$, so the p_i can be taken to be actual slices ($s_i = p_i$). Using the fact that $s_i \in A_i$ we obtain that $A = A_1 = A_2[s_1] = A_3[s_2, s_1] = \cdots = A_{n+1}[s_1, \ldots, s_n] = k[s_1, \ldots, s_n, f]$ as claimed. $\square$

5. Consequences of the main theorems

This paper is originally motivated by the following result of [7]:

Theorem 4. *Let $A = k[x, y, z]$ and D_1, D_2 be two commuting locally nilpotent derivations on A which are linearly independent over A. Then $A^{D_1, D_2} = k[f]$ and f is a coordinate.*

Here the notation A^{D_1, D_2} means $A^{D_1} \cap A^{D_2}$ the intersections of the kernels of D_1 and D_2, which is the set of elements vanishing under D_1 resp. D_2. (Note that for the $\mathbb{G}_a$ action associated to D, this notation means $\mathcal{O}(X/\mathbb{G}_a) = \mathcal{O}(X)^{\mathbb{G}_a} = \mathcal{O}(X)^D$.) By a **coordinate** is meant an element f in $k^{[n]}$ for which there exist $f_2, \ldots, f_n$ with $k[f, f_2, \ldots, f_n] = k^{[n]}$. Equivalently, $(f, f_2, \ldots, f_n) : k^{[n]} \to k^{[n]}$ is an automorphism. The most important ingredient in the proof of this theorem is Kaliman's theorem [6].

In [7] it is conjectured that this result is true also in higher dimensions, namely,

***CDC*$(\boldsymbol{n})$ Commuting Derivations Conjecture.** *The common kernel of n commuting linearly independent locally nilpotent derivations of $k^{[n+1]}$ is generated by a coordinate.*

It seems that this conjecture is difficult, on a par with the well-known conjecture by Sathaye:

***SC*$(\boldsymbol{n})$ Sathaye Conjecture.** *A polynomial $f \in A := k^{[n]}$ for which $A/(f - \lambda) \cong k^{[n-1]}$ for all $\lambda \in k$ is a coordinate.*

The Sathaye conjecture is proved for $n \leqslant 3$ by the aforementioned Kaliman's theorem. The original motivation for this paper was to find additional restrictions in higher dimensions that would achieve at least a partial proof of *CDC*(n). One such requirement is given in Theorem 2, namely that $k^{[n]}/(f - \lambda) \cong k^{[n-1]}$ for all constants λ. A closer examination reveals an interesting equivalent reformulation of the Sathaye conjecture:

***MSC*$(\boldsymbol{n})$ Modified Sathaye Conjecture.** *Let $A := k^{[n]}$, and let $f \in A$ be such that $A/(f - \alpha) \cong k^{[n-1]}$ for all $\alpha \in k$. Then there exist $n - 1$ commuting locally nilpotent derivations $D_1, \ldots, D_{n-1}$ on A such that $A^{D_1, \ldots, D_{n-1}} = k[f]$ and the D_i are linearly independent modulo $(f - \alpha)$ for each $\alpha \in k$.*

Proof of equivalence of SC$(\boldsymbol{n})$ and MSC$(\boldsymbol{n})$. Suppose we have proven the *MSC*(n). Then for any f satisfying "$A/(f - \alpha) \cong k^{[n-1]}$ for all $\alpha \in k$" we can find commuting LNDs $D_1, \ldots, D_{n-1}$ on A giving rise to a $\mathbb{G}_a^{n-1}$ action satisfying the hypotheses of Theorem 2. Applying this theorem, we obtain that f is a coordinate in A. So the *SC*(n) is true in that case.

Now suppose we have proven the $SC(n)$. Let f satisfy the requirements of the $MSC(n)$, that is, "$A/(f-\alpha) \cong k^{[n-1]}$ for all $\alpha \in k$". Since f satisfies the requirements of $SC(n)$, f then must be a coordinate. So it has $n-1$ so-called mates: $k[f, f_2, \dots, f_n] = k^{[n]}$. But then the partial derivative with respect to each of these n polynomials $f, f_2, \dots, f_n$ defines a locally nilpotent derivation. All of them commute, and the intersection of the kernels of the last $n-1$ derivations is $k[f]$; so the MSC holds. $\square$

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